

# Adaptive Synchronization of An Uncertain Complex Dynamical Network

Jin Zhou, Junan Lu, and Jinhu Lü

**Abstract**—This brief paper further investigates the locally and globally adaptive synchronization of an uncertain complex dynamical network. Several network synchronization criteria are deduced. Especially, our hypotheses and designed adaptive controllers for network synchronization are rather simple in form. It is very useful for future practical engineering design. Moreover, numerical simulations are also given to show the effectiveness of our synchronization approaches.

**Index Terms**—Complex networks, adaptive synchronization, uncertain systems

## I. INTRODUCTION

OVER THE PAST DECADE, complex networks have been intensively studied in various disciplines, such as social, biological, mathematical, and engineering sciences [1-8]. A complex network is a large set of interconnected nodes, where the nodes and connections can be anything. Detailed examples are the World Wide Web, Internet, communication networks, metabolic systems, food webs, electrical power grids, and so on.

Recently, one of the interesting and significant phenomena in complex dynamical networks is the synchronization of all dynamical nodes in a network. In fact, synchronization is a kind of typical collective behaviors and basic motions in nature. For example, the synchronization of coupled oscillators can explain well many natural phenomena. Furthermore, some synchronization phenomena are very useful in our daily life, such as the synchronous transfer of digital or analog signals in communication networks. Specifically, synchronization in networks of coupled chaotic systems has received a great deal of attention. Some synchronization criteria of two or three Lorenz systems have been obtained in the literature. However, it is often difficult to get the exact estimation of the coupling coefficients since we do not know the exact boundary for most chaotic systems. Up to now, we can only estimate the boundary of very few chaotic systems [9-13], such as the Lorenz, Chen, Lü systems [14]. Moreover, we often know very little information on the network structure, which makes network design very difficult. To overcome these difficulties, an effectively adaptive synchronization approach is proposed

based on an uncertain complex dynamical network model in this paper.

Slotine, Wang, and Rifai [16,17] further discussed the synchronization of nonlinearly coupled continuous and hybrid oscillators networks by using the contraction analysis approach [18]. Bohacek and Jonckheere [19-20] proposed the so-called linear dynamically varying method based on discrete time dynamical systems. In the following, by using Lyapunov stability theory, several novel locally and globally asymptotically stable network synchronization criteria are deduced for an uncertain complex dynamical network. Compared with some similar results [3,5,15], our sufficient conditions for network synchronization are rather broad and the controllers are very simple. It is very useful for future practical engineering design. Moreover, our analysis method and network model are very different from those of the above referenced literature [16-20]. However, for some complex systems (e.g., biological systems) with unknown couplings, our conditions are hard to be verified. In fact, it is impossible to propose a universal synchronization criterion for various complex networks since there are many uncertain factors, such as network structures and coupling mechanisms.

This paper is organized as follows. An uncertain complex dynamical network model and several necessary hypotheses are given in Section II. In Section III, locally and globally adaptive synchronization criteria for uncertain complex dynamical networks are proposed. In Section IV, a simple example is provided to verify the effectiveness of the proposed method. Finally, conclusions are given in Section V.

## II. PRELIMINARIES

This section introduces an uncertain complex dynamical network model and gives some preliminary definitions and hypotheses.

### A. An uncertain complex dynamical network model

Consider an uncertain complex dynamical network consisting of  $N$  identical nonlinear oscillators with uncertain nonlinear diffusive couplings, which is described by

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, t) + \mathbf{h}_i(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N) + \mathbf{u}_i, \quad (1)$$

where  $1 \leq i \leq N$ ,  $\mathbf{x}_i = (x_{i1}, x_{i2}, \dots, x_{in})^T \in \mathbf{R}^n$  is the state vector of the  $i$ th node,  $\mathbf{f} : \Omega \times \mathbf{R}^+ \rightarrow \mathbf{R}^n$  is a smooth nonlinear vector field, node dynamics is  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ ,  $\mathbf{h}_i : \Omega \times \dots \times \Omega \rightarrow \mathbf{R}^n$  are unknown nonlinear smooth diffusive coupling functions,  $\mathbf{u}_i \in \mathbf{R}^n$  are the control inputs, and the coupling-control terms satisfy  $\mathbf{h}_i(\mathbf{s}, \mathbf{s}, \dots, \mathbf{s}) + \mathbf{u}_i = \mathbf{0}$ , where  $\mathbf{s}$  is a synchronous solution of the node system  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ .

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### B. Preliminaries

Network synchronization is a typical collective behavior. In the following, a rigorous mathematical definition is introduced for the concept of network synchronization.

**Definition 1:** Let  $\mathbf{x}_i(t; t_0, \mathbf{X}_0)$  ( $1 \leq i \leq N$ ) be a solution of the dynamical network (1), where  $\mathbf{X}_0 = (\mathbf{x}_1^0, \mathbf{x}_2^0, \dots, \mathbf{x}_N^0)$ ,  $\mathbf{f} : \Omega \times \mathbf{R}^+ \rightarrow \mathbf{R}^n$ , and  $\mathbf{h}_i : \Omega \times \dots \times \Omega \rightarrow \mathbf{R}^n$  ( $1 \leq i \leq N$ ) are continuously differentiable,  $\Omega \subseteq \mathbf{R}^n$ . If there is a nonempty subset  $\Lambda \subseteq \Omega$ , with  $\mathbf{x}_i^0 \in \Lambda$  ( $1 \leq i \leq N$ ), such that  $\mathbf{x}_i(t; t_0, \mathbf{X}_0) \in \Omega$  for all  $t \geq t_0$ ,  $1 \leq i \leq N$ , and

$$\lim_{t \rightarrow \infty} \|\mathbf{x}_i(t; t_0, \mathbf{X}_0) - \mathbf{s}(t; t_0, \mathbf{x}_0)\|_2 = \mathbf{0} \quad 1 \leq i \leq N \quad (2)$$

where  $\mathbf{s}(t; t_0, \mathbf{x}_0)$  is a solution of the system  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$  with  $\mathbf{x}_0 \in \Omega$ , then the dynamical network (1) is said to realize *synchronization* and  $\Lambda \times \dots \times \Lambda$  is called the *region of synchrony* for the dynamical network (1).

Hereafter, denote  $\mathbf{s}(t; t_0, \mathbf{x}_0)$  as  $\mathbf{s}(t)$ . Then  $\mathbf{S}(t) = (\mathbf{s}^T(t), \mathbf{s}^T(t), \dots, \mathbf{s}^T(t))^T$  is a synchronous solution of uncertain dynamical network (1) since it is a diffusive coupling network. Here,  $\mathbf{s}(t)$  can be an equilibrium point, a periodic orbit, an aperiodic orbit, or a chaotic orbit in the phase space.

Define error vector

$$\mathbf{e}_i(t) = \mathbf{x}_i(t) - \mathbf{s}(t), \quad 1 \leq i \leq N. \quad (3)$$

Then the objective of controller  $\mathbf{u}_i$  is to guide the dynamical network (1) to synchronize. That is,

$$\lim_{t \rightarrow +\infty} \|\mathbf{e}_i(t)\|_2 = 0, \quad 1 \leq i \leq N. \quad (4)$$

Since  $\dot{\mathbf{s}} = \mathbf{f}(\mathbf{s}, t)$ , from network (1), we have

$$\dot{\mathbf{e}}_i = \bar{\mathbf{f}}(\mathbf{x}_i, \mathbf{s}, t) + \bar{\mathbf{h}}_i(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N, \mathbf{s}) + \mathbf{u}_i, \quad (5)$$

where  $1 \leq i \leq N$ ,  $\bar{\mathbf{f}}(\mathbf{x}_i, \mathbf{s}, t) = \mathbf{f}(\mathbf{x}_i, t) - \mathbf{f}(\mathbf{s}, t)$ ,  $\bar{\mathbf{h}}_i(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N, \mathbf{s}) = \mathbf{h}_i(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N) - \mathbf{h}_i(\mathbf{s}, \mathbf{s}, \dots, \mathbf{s})$ .

In the following, we give several useful hypotheses.

**Hypothesis 1:** (H1) Assume that there exists a nonnegative constant  $\alpha$  satisfying  $\|\mathbf{Df}(\mathbf{s}, t)\|_2 = \|\mathbf{A}(t)\|_2 \leq \alpha$ , where  $\mathbf{A}(t)$  is the Jacobian of  $\mathbf{f}(\mathbf{s}, t)$ .

**Hypothesis 2:** (H2) Suppose that there exist nonnegative constants  $\gamma_{ij}$  ( $1 \leq i, j \leq N$ ) satisfying  $\|\bar{\mathbf{h}}_i(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N, \mathbf{s})\|_2 \leq \sum_{j=1}^N \gamma_{ij} \|\mathbf{e}_j\|_2$  for  $1 \leq i \leq N$ .

**Remark 1:** If H1 holds, then we get  $\left\| \frac{\mathbf{A}(t) + \mathbf{A}^T(t)}{2} \right\|_2 \leq \alpha$ .

### III. ADAPTIVE SYNCHRONIZATION OF AN UNCERTAIN COMPLEX DYNAMICAL NETWORK

This section discusses the local synchronization and global synchronization of the uncertain complex dynamical network (1). Several network synchronization criteria are given.

#### A. Local Synchronization

Linearizing error system (5) around zero gives

$$\dot{\mathbf{e}}_i = \mathbf{A}(t)\mathbf{e}_i(t) + \bar{\mathbf{h}}_i(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N, \mathbf{s}) + \mathbf{u}_i, \quad (6)$$

where  $1 \leq i \leq N$  and recall that  $\mathbf{A}(t) = \mathbf{Df}(\mathbf{s}, t)$  is the Jacobian of  $\mathbf{f}$  evaluated at  $\mathbf{x} = \mathbf{s}(t)$ .

Based on H1 and H2, a network synchronization criterion is deduced as follows.

**Theorem 1:** Suppose that H1 and H2 hold. Then the synchronous solution  $\mathbf{S}(t)$  of uncertain dynamical network (1) is locally asymptotically stable under the adaptive controllers

$$\mathbf{u}_i = -\hat{d}_i \mathbf{e}_i, \quad 1 \leq i \leq N \quad (7)$$

and updating laws

$$\dot{\hat{d}}_i = k_i \mathbf{e}_i^T \mathbf{e}_i = k_i \|\mathbf{e}_i\|_2^2, \quad 1 \leq i \leq N, \quad (8)$$

where  $k_i$  ( $1 \leq i \leq N$ ) are positive constants.

**Proof:** Define a Lyapunov candidate as follows:

$$V = \frac{1}{2} \sum_{i=1}^N \mathbf{e}_i^T \mathbf{e}_i + \frac{1}{2} \sum_{i=1}^N \frac{(d_i - \hat{d}_i)^2}{k_i}, \quad (9)$$

where  $\hat{d}_i$  ( $1 \leq i \leq N$ ) are positive constants to be determined. Thus one gets

$$\begin{aligned} \dot{V} &= \frac{1}{2} \sum_{i=1}^N (\dot{\mathbf{e}}_i^T \mathbf{e}_i + \mathbf{e}_i^T \dot{\mathbf{e}}_i) + \sum_{i=1}^N \frac{(d_i - \hat{d}_i) \dot{\hat{d}}_i}{k_i} \\ &= \sum_{i=1}^N \mathbf{e}_i^T \left( \frac{\mathbf{A}(t) + \mathbf{A}^T(t)}{2} - d_i \mathbf{I}_n \right) \mathbf{e}_i \\ &\quad + \sum_{i=1}^N \mathbf{e}_i^T \bar{\mathbf{h}}_i(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N, \mathbf{s}) + \sum_{i=1}^N (d_i - \hat{d}_i) \mathbf{e}_i^T \mathbf{e}_i \\ &\leq \sum_{i=1}^N \mathbf{e}_i^T \left( \frac{\mathbf{A}(t) + \mathbf{A}^T(t)}{2} - \hat{d}_i \mathbf{I}_n \right) \mathbf{e}_i \\ &\quad + \sum_{i=1}^N \sum_{j=1}^N \gamma_{ij} \|\mathbf{e}_i\|_2 \|\mathbf{e}_j\|_2 \\ &\leq \sum_{i=1}^N (\alpha - \hat{d}_i) \|\mathbf{e}_i\|_2^2 + \sum_{i=1}^N \sum_{j=1}^N \gamma_{ij} \|\mathbf{e}_i\|_2 \|\mathbf{e}_j\|_2 \\ &= \mathbf{e}^T (\mathbf{\Gamma} + \text{diag}\{\alpha - \hat{d}_1, \alpha - \hat{d}_2, \dots, \alpha - \hat{d}_N\}) \mathbf{e}, \end{aligned}$$

where  $\mathbf{e} = (\|\mathbf{e}_1\|_2, \|\mathbf{e}_2\|_2, \dots, \|\mathbf{e}_N\|_2)^T$  and  $\mathbf{\Gamma} = (\gamma_{ij})_{N \times N}$ .

Since  $\alpha$  and  $\gamma_{ij}$  ( $1 \leq i, j \leq N$ ) are nonnegative constants, one can select suitable constants  $\hat{d}_i$  ( $1 \leq i \leq N$ ) to make  $\mathbf{\Gamma} + \text{diag}\{\alpha - \hat{d}_1, \alpha - \hat{d}_2, \dots, \alpha - \hat{d}_N\}$  a negative definite matrix. Thus it follows that the error vector  $\eta = (\mathbf{e}_1^T, \mathbf{e}_2^T, \dots, \mathbf{e}_N^T)^T \rightarrow \mathbf{0}$  as  $t \rightarrow +\infty$ . That is, the synchronous solution  $\mathbf{S}(t)$  of uncertain dynamical network (1) is locally asymptotically stable under the adaptive controllers (7) and updating laws (8).

The proof is thus completed.

Assume that the coupling of network (1) is linear satisfying  $\mathbf{h}_i(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N) = \sum_{j=1}^N b_{ij} \mathbf{x}_j$  for  $1 \leq i \leq N$ , where  $b_{ij}$  ( $1 \leq i, j \leq N$ ) are constants. Then the uncertain network (1) is recasted as follows:

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, t) + \sum_{j=1}^N b_{ij} \mathbf{x}_j + \mathbf{u}_i, \quad 1 \leq i \leq N. \quad (10)$$

For linear coupling, H2 is naturally satisfied. Thus one gets the following corollaries.

*Corollary 1:* Suppose that H1 holds. Then the synchronous solution  $\mathbf{S}(t)$  of the uncertain dynamical network (10) is locally asymptotically stable under the adaptive controllers (7) and updating laws (8).

Moreover, for the coupling scheme  $\mathbf{h}_i(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N) = \sum_{j=1}^N b_{ij} \mathbf{p}(\mathbf{x}_j)$  with  $1 \leq i \leq N$ , where  $b_{ij}$  ( $1 \leq i, j \leq N$ ) are constants satisfying  $\sum_{j=1}^N b_{ij} = 0$  for  $1 \leq i \leq N$  and  $\|\mathbf{Dp}(\xi)\|_2 \leq \delta$  for  $\xi \in \Omega$ , the network (1) is rewritten as follows:

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, t) + \sum_{j=1}^N b_{ij} \mathbf{p}(\mathbf{x}_j) + \mathbf{u}_i, \quad 1 \leq i \leq N. \quad (11)$$

If H1 holds, then one has

$$\begin{aligned} \|\bar{\mathbf{h}}_i(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N, \mathbf{s})\|_2 &= \sum_{j=1}^N |b_{ij}| \|\mathbf{p}(\mathbf{x}_j) - \mathbf{p}(\mathbf{s})\|_2 \\ &\leq \sum_{j=1}^N \delta |b_{ij}| \|\mathbf{e}_j\|_2 \end{aligned}$$

for  $1 \leq i \leq N$ . That is, H2 holds and one gets the following corollary.

*Corollary 2:* Assume that H1 holds. Then, under the adaptive controllers (7) and updating laws (8), the synchronous solution  $\mathbf{S}(t)$  of the uncertain dynamical network (11) is locally asymptotically stable.

In the following subsection, we discuss the global synchronization case.

### B. Global synchronization

This section presents two global network synchronization criteria.

Rewrite node dynamics  $\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, t)$  as  $\dot{\mathbf{x}}_i = \mathbf{B}\mathbf{x}_i(t) + \mathbf{g}(\mathbf{x}_i, t)$ , where  $\mathbf{B} \in \mathbf{R}^{n \times n}$  is a constant matrix and  $\mathbf{g} : \Omega \times \mathbf{R}^+ \rightarrow \mathbf{R}^n$  is a smooth nonlinear function. Thus network (1) is described by

$$\dot{\mathbf{x}}_i = \mathbf{B}\mathbf{x}_i(t) + \mathbf{g}(\mathbf{x}_i, t) + \mathbf{h}_i(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N) + \mathbf{u}_i, \quad (12)$$

where  $1 \leq i \leq N$ . Similarly, one can get the error system

$$\dot{\mathbf{e}}_i = \mathbf{B}\mathbf{e}_i(t) + \bar{\mathbf{g}}(\mathbf{x}_i, \mathbf{s}, t) + \bar{\mathbf{h}}_i(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N, \mathbf{s}) + \mathbf{u}_i, \quad (13)$$

where  $1 \leq i \leq N$  and  $\bar{\mathbf{g}}(\mathbf{x}_i, \mathbf{s}, t) = \mathbf{g}(\mathbf{x}_i, t) - \mathbf{g}(\mathbf{s}, t)$ .

*Hypothesis 3:* (H3) Suppose that there exists a nonnegative constant  $\mu$  satisfying  $\|\bar{\mathbf{g}}(\mathbf{x}_i, \mathbf{s}, t)\|_2 \leq \mu \|\mathbf{e}_i\|_2$ .

Then one can get the following global network synchronization criterion.

*Theorem 2:* Suppose that H2 and H3 hold. Then the synchronous solution  $\mathbf{S}(t)$  of uncertain dynamical network (1) is globally asymptotically stable under the adaptive controllers

$$\mathbf{u}_i = -d_i \mathbf{e}_i, \quad 1 \leq i \leq N \quad (14)$$

and updating laws

$$\dot{d}_i = k_i \mathbf{e}_i^T \mathbf{e}_i = k_i \|\mathbf{e}_i\|_2^2, \quad 1 \leq i \leq N, \quad (15)$$

where  $k_i$  ( $1 \leq i \leq N$ ) are positive constants.

*Proof:* Since  $\mathbf{B}$  is a given constant matrix, there exists a nonnegative constant  $\beta$  such that  $\|\mathbf{B}\|_2 \leq \beta$ . It follows that  $\left\| \frac{\mathbf{B} + \mathbf{B}^T}{2} \right\|_2 \leq \beta$ .

Similarly, construct Lyapunov function (9), then one has

$$\begin{aligned} \dot{V} &= \sum_{i=1}^N \mathbf{e}_i^T \left( \frac{\mathbf{B} + \mathbf{B}^T}{2} - \hat{d}_i \mathbf{I}_n \right) \mathbf{e}_i + \sum_{i=1}^N \mathbf{e}_i^T \bar{\mathbf{g}}_i(\mathbf{x}_i, \mathbf{s}, t) \\ &\quad + \sum_{i=1}^N \mathbf{e}_i^T \bar{\mathbf{h}}_i(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N, \mathbf{s}) \\ &\leq \sum_{i=1}^N (\beta + \mu - \hat{d}_i) \|\mathbf{e}_i\|_2^2 + \sum_{i=1}^N \sum_{j=1}^N \gamma_{ij} \|\mathbf{e}_i\|_2 \|\mathbf{e}_j\|_2 \\ &= \mathbf{e}^T (\mathbf{\Gamma} + \text{diag}\{\beta + \mu - \hat{d}_1, \dots, \beta + \mu - \hat{d}_N\}) \mathbf{e}, \end{aligned}$$

where  $\mathbf{e} = (\|\mathbf{e}_1\|_2, \|\mathbf{e}_2\|_2, \dots, \|\mathbf{e}_N\|_2)^T$  and  $\mathbf{\Gamma} = (\gamma_{ij})_{N \times N}$ .

Since  $\beta$ ,  $\mu$  and  $\gamma_{ij}$  ( $1 \leq i, j \leq N$ ) are nonnegative constants, one can select suitable constants  $\hat{d}_i$  ( $1 \leq i \leq N$ ) to make  $\mathbf{\Gamma} + \text{diag}\{\beta + \mu - \hat{d}_1, \beta + \mu - \hat{d}_2, \dots, \beta + \mu - \hat{d}_N\}$  a negative definite matrix. Then the error vector  $\boldsymbol{\eta} = (\mathbf{e}_1^T, \mathbf{e}_2^T, \dots, \mathbf{e}_N^T)^T \rightarrow \mathbf{0}$  as  $t \rightarrow +\infty$ . That is, the synchronous solution  $\mathbf{S}(t)$  of uncertain dynamical network (1) is globally asymptotically stable under the adaptive controllers (14) and updating laws (15).

This completes the proof.

Similarly, one gets the following two corollaries of global network synchronization.

*Corollary 3:* Suppose that H3 holds. Then the synchronous solution  $\mathbf{S}(t)$  of uncertain linearly coupled dynamical network (10) is globally asymptotically stable under the adaptive controllers (14) and updating laws (15).

*Corollary 4:* Suppose that H3 holds. Then the synchronous solution  $\mathbf{S}(t)$  of uncertain dynamical network (11) is globally asymptotically stable under the adaptive controllers (14) and updating laws (15).

*Proof.* According to (11), one has

$$\begin{aligned} \|\bar{\mathbf{h}}_i(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N, \mathbf{s})\|_2 &= \left\| \sum_{j=1}^N b_{ij} (\mathbf{B}\mathbf{e}_j + \bar{\mathbf{g}}_j(\mathbf{x}_j, \mathbf{s}, t)) \right\|_2 \\ &\leq \sum_{j=1}^N b_{ij} (\|\mathbf{B}\| + \mu) \|\mathbf{e}_j\|; \end{aligned}$$

thus H2 holds. Therefore, from Theorem 2, the synchronous solution  $\mathbf{S}(t)$  of network (11) is globally asymptotically stable under the adaptive controllers (14) and updating laws (15).

The proof is thus completed.

*Hypothesis 4:* (H4) Assume that  $\mathbf{g}(\mathbf{x}, t)$  satisfies the Lipschitz condition. That is, there exists a positive constant  $\kappa$  satisfying  $\|\mathbf{g}(\mathbf{x}, t) - \mathbf{g}(\mathbf{y}, t)\| \leq \kappa \|\mathbf{x} - \mathbf{y}\|$ , where  $\kappa$  is the Lipschitz constant.

Obviously, H4 implies H3. Now one has the following synchronization criterion.

*Theorem 3:* Suppose that H2 and H4 hold. Then the synchronous solution  $\mathbf{S}(t)$  of uncertain dynamical network (1)

is globally asymptotically stable under the adaptive controllers (14) and updating laws (15).

#### IV. AN EXAMPLE

This section presents an example to show the effectiveness of the above synchronization criteria.

Consider a dynamical network consisting of 50 identical Lorenz systems. Here, node dynamics is described by

$$\begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{pmatrix} + \begin{pmatrix} 0 \\ -x_{i1}x_{i3} \\ x_{i1}x_{i2} \end{pmatrix},$$

where

$$\mathbf{A} = \begin{pmatrix} -a & a & 0 \\ c & -1 & 0 \\ 0 & 0 & -b \end{pmatrix},$$

$a = 10, b = \frac{8}{3}, c = 28$  and  $1 \leq i \leq 50$ . And the networked system is defined as follows:

$$\begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{pmatrix} + \begin{pmatrix} 0 \\ -x_{i1}x_{i3} \\ x_{i1}x_{i2} \end{pmatrix} + \begin{pmatrix} f_1(\mathbf{x}_i) - 2f_1(\mathbf{x}_{i+1}) + f_1(\mathbf{x}_{i+2}) \\ 0 \\ f_2(\mathbf{x}_i) - 2f_2(\mathbf{x}_{i+1}) + f_2(\mathbf{x}_{i+2}) \end{pmatrix} + d_i \mathbf{e}_i \quad (16)$$

and

$$\dot{d}_i = k_i \|\mathbf{e}_i\|_2^2, \quad (17)$$

$f_1(\mathbf{x}_i) = a(x_{i2} - x_{i1}), f_2(\mathbf{x}_i) = x_{i1}x_{i2} - bx_{i3}$ ,  $x_{51} \equiv x_1, x_{52} \equiv x_2$ , and  $1 \leq i \leq 50$ .

Obviously, one gets

$$\begin{aligned} \bar{\mathbf{g}}(\mathbf{x}_i, s, t) &= \begin{pmatrix} 0 \\ -x_{i1}x_{i3} + s_1s_3 \\ x_{i1}x_{i2} - s_1s_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -x_{i3}e_{i1} - s_1e_{i3} \\ x_{i2}e_{i1} + s_1e_{i2} \end{pmatrix}, \end{aligned}$$

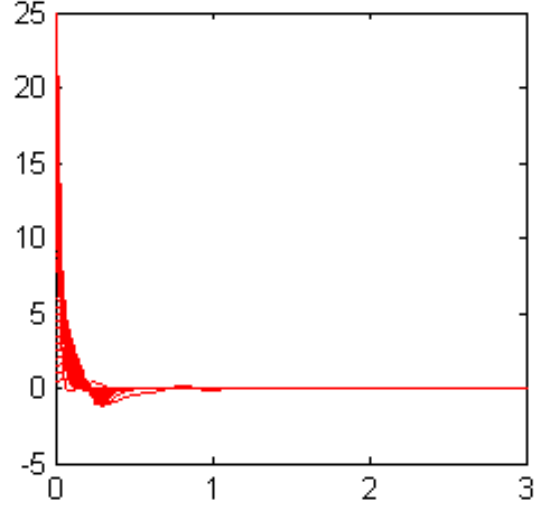
where  $1 \leq i \leq 50$ .

Since Lorenz attractor is confined to a bounded region  $\Phi \subset \mathbf{R}^3$  [9-13], there exists a constant  $M$  satisfying  $|x_{ij}|, |s_j| \leq M$  for  $1 \leq i \leq 50$  and  $j = 1, 2, 3$ . Therefore,  $\|\bar{\mathbf{g}}(\mathbf{x}_i, s, t)\|_2 = \sqrt{(x_{i3}e_{i1} + s_1e_{i3})^2 + (x_{i2}e_{i1} + s_1e_{i2})^2} \leq 2M\|\mathbf{e}_i\|_2$ .

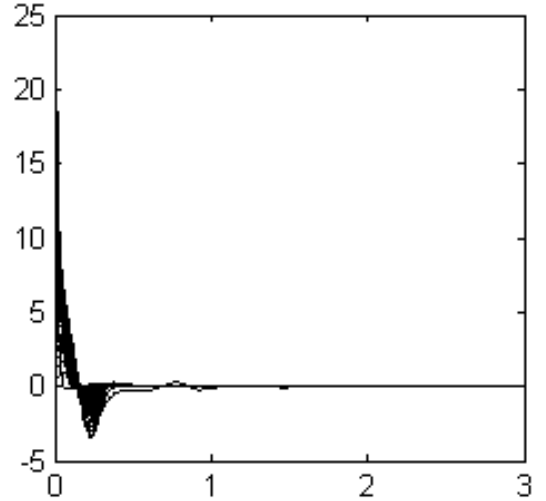
Similarly, one has

$$\bar{\mathbf{h}}_i(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N, \mathbf{s}) = \begin{pmatrix} f_1(\mathbf{e}_i) - 2f_1(\mathbf{e}_{i+1}) + f_1(\mathbf{e}_{i+2}) \\ 0 \\ f_3(\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}, \mathbf{s}) \end{pmatrix},$$

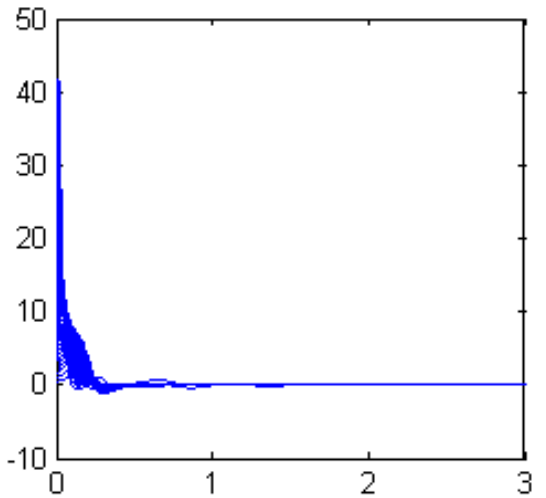
where  $f_3(\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}, \mathbf{s}) = -be_{i3} + 2be_{i+1,3} - be_{i+2,3} + x_{i2}e_{i1} + s_1e_{i2} - 2(x_{i+1,2}e_{i+1,1} + s_1e_{i+1,2}) + x_{i+2,2}e_{i+2,1} + s_1e_{i+2,2}$  and  $1 \leq i \leq 50$ .



(a)  $\mathbf{e}_{i1}$  ( $1 \leq i \leq 50$ )



(b)  $\mathbf{e}_{i2}$  ( $1 \leq i \leq 50$ )



(c)  $\mathbf{e}_{i3}$  ( $1 \leq i \leq 50$ )

Fig. 1. Synchronization errors of network (16)-(17).

Since

$$\begin{aligned}
& \|\bar{\mathbf{h}}_i(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N, \mathbf{s})\|_2^2 \\
&= (f_1(\mathbf{e}_i) - 2f_1(\mathbf{e}_{i+1}) + f_1(\mathbf{e}_{i+2}))^2 \\
&\quad + (f_3(\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}, \mathbf{s}))^2 \\
&\leq (a\|\mathbf{e}_i\|_1 + 2a\|\mathbf{e}_{i+1}\|_1 + a\|\mathbf{e}_{i+2}\|_1)^2 \\
&\quad + (M\|\mathbf{e}_i\|_1 + 2M\|\mathbf{e}_{i+1}\|_1 + M\|\mathbf{e}_{i+2}\|_1)^2 \\
&\leq 6(a^2 + M^2)(\|\mathbf{e}_i\|_1^2 + \|\mathbf{e}_{i+1}\|_1^2 + \|\mathbf{e}_{i+2}\|_1^2) \\
&\leq 18(a^2 + M^2)(\|\mathbf{e}_i\|_2^2 + \|\mathbf{e}_{i+1}\|_2^2 + \|\mathbf{e}_{i+2}\|_2^2) \\
&\leq 18(a^2 + M^2)(\|\mathbf{e}_i\|_2 + \|\mathbf{e}_{i+1}\|_2 + \|\mathbf{e}_{i+2}\|_2)^2,
\end{aligned}$$

where  $1 \leq i \leq 50$ , one gets

$$\begin{aligned}
& \|\bar{\mathbf{h}}_i(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N, \mathbf{s})\|_2 \\
&\leq 3\sqrt{2(a^2 + M^2)}(\|\mathbf{e}_i\|_2 + \|\mathbf{e}_{i+1}\|_2 + \|\mathbf{e}_{i+2}\|_2).
\end{aligned}$$

Thus H2 and H3 hold. According to Theorem 2, the synchronous solution  $\mathbf{S}(t)$  of dynamical network (16)-(17) is globally asymptotically stable.

Assume that  $k_i = 1$ ,  $d_i(0) = 1$ ,  $\mathbf{x}_i(0) = (4 + 0.5i, 5 + 0.5i, 6 + 0.5i)$  for  $1 \leq i \leq 50$  and  $\mathbf{s}(0) = (4, 5, 6)$ . The synchronous error  $\mathbf{e}_i$  is shown in Fig. 1. Obviously, the zero error is globally asymptotically stable for dynamical network (16)-(17).

*Remark 2:* It is well known that the nearest-neighbor coupled ring lattices are very hard to synchronize. This is because the coupling coefficient  $c$  satisfies  $c = O(N^2)$ . However, the above example shows that the synchronization of nearest-neighbor coupled ring lattice will be relatively easy by adding a simple adaptive controller.

## V. CONCLUSIONS

We have further studied the locally and globally adaptive synchronization of an uncertain complex dynamical network. Several novel network synchronization criteria have been proved by using Lyapunov stability theory. Compared with some similar results, our assumptions and adaptive controllers are very simple. Furthermore, the effectiveness of these synchronization criteria have been demonstrated by numerical simulations.

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